Eulerian Polynomials and B-Splines

Tian-Xiao He
Department of Mathematics and Computer Science
Illinois Wesleyan University
Bloomington, IL 61702-2900, USA

Abstract

Here presented is the interrelationship between Eulerian polynomials, Eulerian fractions and Euler-Frobenius polynomials, Euler-Frobenius fractions, B-splines, respectively. The properties of Eulerian polynomials and Eulerian fractions and their applications in B-spline interpolation and evaluation of Riemann zeta function values at odd integers are given. The relation between Eulerian numbers and B-spline values at knot points are also discussed.

AMS Subject Classification: 41A80, 65B10, 33C45, 39A70.

Key Words and Phrases: B-spline, Eulerian polynomial, Eulerian Number, Eulerian Fraction, exponential Euler polynomial, Euler-Frobenius polynomial, and Euler-Frobenius fractions.

1 Introduction

Eulerian polynomial sequence $\{A_n(z)\}_{n\geq 0}$ is defined by the following summation (*cf.* for examples, [1], [2], and [3]):

$$\sum_{\ell\geq 0} \ell^n z^\ell = \frac{A_n(z)}{(1-z)^{n+1}}, \quad |z|<1. \tag{1.1}$$

It is well-known that the Eulerian polynomial, $A_n(z)$ (see p. 244 of [3]), of degree n can be written in the form

$$A_n(z) = \sum_{k=1}^{n} A(n,k) z^k, \quad A_0(z)=1, \tag{1.2}$$

where $A(n,k)$ are called the *Eulerian numbers* that can be calculated by using

$$A(n,k) = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j)^n, \quad 1\leq k\leq n, \tag{1.3}$$

and $A(n,0)=1$. The Eulerian numbers are found, for example, in the standard treatises by Riordan [4], Comtet [3], Aigner [5], Grahm et al. [6], etc. However, their notations are not standard, which are shown by $A_{n,k}$, $A(n,k)$, $W_{n,k}$, $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$, and $E(n,k)$, respectively. The Eulerian number $A(n,k)$ gives the number of permutations in the symmetric group of order n that have exactly $k-1$ ascents, or equivalently, the number of permutation runs of length $k-1$ (*cf.* [3]), where in an ascending sequence of permutation, j is a permutation ascent if the jth term in the sequence less than the $j+1$st term. A geometric interpretation of $A(n,k)$ is given in [8]. One may re-write (1.3) as

$$A(n,k)=\sum_{j=0}^{n}(-1)^j\binom{n+1}{j}(k-j)_+^n, \quad 1\le k\le n,$$

where $(a)_+=\max\{0,a\}$, which provides an interrelation between Eulerian numbers and B-spline values at knot points due to the expression of B-spline shown below in (1.5). It is well-known that the Eulerian fraction is a powerful tool in the study of the Eulerian numbers, Eulerian polynomial, Euler function and its generalization, Jordan function, number theory, etc. (*cf.* [9]). The classical *Eulerian fraction*, $\alpha_n(x)$, can be expressed in the form

$$\alpha_n(z):=\frac{A_n(z)}{(1-z)^{n+1}}, \quad z\ne 1. \tag{1.4}$$

Polynomial spline functions can be considered as broken polynomials with certain smoothness, which are used to overcome the stiffness of polynomials, for instance the Runge phenomenon in the polynomial interpolation. *B spline functions* are probably the most applicable one-dimensional polynomial spline functions, where *B-spline*, denoted by $M(x)\equiv M(x;x_0,\ldots,x_n)$ $(x_0<x_x<\cdots<x_n)$, is defined by

$$M(x;x_0,\ldots,x_n)=\sum_{j=0}^{n}\frac{n(x_j-x)_+^{n-1}}{\omega'(x_j)},\ x\in\mathbb{R}, \tag{1.5}$$

where $\omega(x)=(x-x_0)\cdots(x-x_n)$ and $x_+=\max\{0,x\}$ (*cf.* [10]). The *nth* order *cardinal forward B-spline* $B_n=M(x;0,1,\ldots,n)$, or simply, B-spline of order n, can be defined by

$$B_n(x)=(B_{n-1}*B_1)(x)=\int_0^1 B_{n-1}(x-t)dt, \quad n\ge 2,$$

where B_1 is the characteristic function of the interval $[0,1)$. It is clear that $B_n(x)\in C^{n-2}$ is a piecewise polynomial of degree $n-1$. One may find more details on the splines and B-splines in [9]-[13] and [14] and some applications in [15] and [16]. In [17], Dyn and Ron considered periodic exponential B-splines defined by weight functions $w_i(u)=e^{a_iu}r_i(u)$ with $r_i(u+1)=cr_i(u)$, $a_i\in\mathbb{R}$, and showed these B-splines possess a significant property of translation invariant and satisfy a generalized Hermite-Genocchi formula. Ron also defined the higher dimensional n-directional exponential box splines in [18].

In this paper, we are more interested in another approach to construct the exponential splines to the base z by using a linear combination of the translations of a polynomial B-spline with combination coefficients z^j . The exponential splines to the base z can be presented and evaluated by using exponential Euler polynomials (EEPs), which connect to some famous generating functions (GF's) in the combinatorics. The details can be found in Lecture 3 of [10] [11], [13], and Chapter 2 of [12]. For the sake of convenience, we shall briey quote them in the next section. This approach also build a strong interrelationship between two different fields of spline approximation and combinatorics. For instance, we shall see the equivalence between Eulerian polynomials and Euler-Frobenius polynomials, Eulerian numbers and B-spline values, Eulerian fractions and Euler-Frobenius fractions, etc. All of those as well as some properties will be presented in Section 2. The applications in spline image interpolation and wavelet analysis and evaluation of Riemann zeta function values at odd integers will be discussed in Section 3.

Only after finishing the paper did we find out that one of our results shown in Corollary 2.7, the relation between the Eulerian numbers and the B-spline values at integers, was obtained by Wang et al. in [20] using a different approach, where the relation was considered as the spline interpretation of Eulerian numbers. For the sake of readers' convenience, we retain the results. In [20], spline interpretation of refined Eulerian numbers was also given. However, the interrelationships among the Eulerian polynomials, spline functions, Eulerian fractions, and Euler-Frobenius polynomials, etc. and their applications are not discussed, which is the main body of this paper.

This paper is prepared partially based on the conference talk notes [19]. The author greatly appreciate the discussion with Carl deBoor since 2005 [21] and the valuable suggestions and the lecture [12] he offered fervently.

2 Eulerian polynomials and B-splines

Let n be a non-negative integer. The symbol S_n denote the class of splines of order n, i.e., functions $s(x)$ satisfying the following conditions: (1) $s(x) \in C^{n-1}(\mathbb{R})$ and (2) $s(x) \in \pi_n$ in each interval $(j, j+1)$, $j = 0, \pm 1, \pm 2, \ldots$, where π_n is the class of polynomials of degree not exceeding n. An exponential spline $f \in S_n$ means an element in S_n satisfying the functional equation

$$f(x+1) = cf(x). \tag{2.1}$$

Let us recall the notations presented in the central part of Schoenberg's lectures on "Cardinal Spline Interpolation" (CSI) (*cf.* [10]). First, the *exponential spline* $\phi_n(x; z)$ *of degree* n *to the base* z is defined by

$$\phi_n(x; z) := \sum_{-\infty}^{\infty} z^j B_{n+1}(x - j),$$

where $B_n(x) := M(x; 0, 1, \cdots, n)$ denotes the cardinal B-spline, or simply, *B-spline* (*cf.* introduction). Obviously, $B_n(x) \in S_{n-1}$. Therefore, $\phi_n \in S_n$, and it is easy to find ϕ_n satisfies (2.1); i.e., $\phi_n(x+1;z) = z\phi_n(x;z)$. In addition, $\phi_n(x;z)$ is a polynomial in the interval $0 < x < 1$ with the form

$$\phi_n(x;z) = \frac{1}{n!}(1-z^{-1})^n x^n + \textit{lower degree terms}$$

for $0 < x < 1$. Hence, $A_n(x;z)$ (Noting $A_n(x;z)$ is not $A_n(x)$) defined by

$$A_n(x;z) := n!(1-z^{-1})^{-n}\phi_n(x;z), \quad 0 \le x \le 1, z \ne 0, 1, \tag{2.2}$$

is a monic polynomial in $0 \le x \le 1$, which is called *the exponential Euler polynomial (EEP) of degree n to the base z*. Furthermore, from [10], the generating function of $\{A_n(x;z)\}$ is

$$\frac{z-1}{z-e^t}e^{xt} = \sum_{n\ge 0} A_n(x;z)\frac{t^n}{n!}. \tag{2.3}$$

If $x = 0$, then (2.3) reduces to

$$\frac{z-1}{z-e^t} = \sum_{n\ge 0} \beta_n(z)\frac{t^n}{n!}, \tag{2.4}$$

where $\beta_n(z) = A_n(0;z)$. It is easy to see that

$$A_n(x;z) = \sum_{j=0}^{n} \binom{n}{j}\beta_j(z)x^{n-j}. \tag{2.5}$$

In particular,

$$A_n(1;z) = \sum_{j=0}^{n} \binom{n}{j}\beta_j(z). \tag{2.6}$$

Hence, we call $\beta_n(z)$ the *Euler-Frobenius fraction* for the reason shown below.

In addition, multiplying (2.4) by $z - e^t$ and comparing the coefficients of the powers of z, we have the relations

$$\beta_0(z) = 1 \quad z\beta_n(z) = \sum_{k=0}^{n} \binom{n}{k}\beta_k(z). \tag{2.7}$$

Jointing (2.6) and (2.7) yields

$$A_n(1;z) = zA_n(0;z). \tag{2.8}$$

Finally, we call

$$\Pi_n(z) := \beta_n(z)(z-1)^n \equiv A_n(0;z)(z-1)^n \tag{2.9}$$

the *Euler-Frobenius polynomials*. Since

$$A_n(0;z) := n!(1-z^{-1})^{-n}\phi_n(0;z) = n!(1-z^{-1})^{-n}\sum_{-\infty}^{\infty} z^j B_{n+1}(-j), \tag{2.10}$$

$\Pi_n(z)$ defined by (2.9) can be written as

$$\Pi_n(z) = n!\sum_{-\infty}^{\infty} B_{n+1}(-j)z^{n+j} = n!\sum_{j=0}^{n-1} B_{n+1}(n-j)z^j = n!\sum_{j=0}^{n-1} B_{n+1}(j+1)z^j. \tag{2.11}$$

From (2.11), we know $\Pi_n(z)$ has $n-1$ zeros, and all of them are negative and simple. In addition, λ is a zero of Π_n if and only if $1/\lambda$ is its zero. The proofs can be found, for example, from [12].

Thus, we have two triples of concepts from combinatorics and spline approximation, respectively, namely, Eulerian polynomials, Eulerian numbers, and Eulerian fractions, $(A_n(z), \alpha(z), A(n,k))$, and Euler-Frobenius polynomials, Euler-Frobenius fractions, and B-spline values, $(\Pi_n(z), \beta_n(z), B_n(k))$, where $n \geq k \geq 0$ We now establish a connection between them, i.e., Eulerian polynomials and , Euler-Frobenius polynomials, Eulerian fractions and the coefficients of Euler-Frobenius polynomials, and Eulerian numbers and discrete B-splines $\{B_n(k)\}_{n\geq k\geq 0}$.

Theorem 2.1 *Let $\Pi_n(z)$ and $\beta_n(z)$ be the polynomials defined by (2.9) and (2.4), respectively, and let Eulerian polynomials $A_n(z)$ and Eulerian fractions $\alpha_n(z)$ be defined by (1.1) and (1.4), respectively. Then we can set the interrelationship between the concepts as*

$$A_n(z) = \begin{cases} \Pi_n(z) = 1 & if\ n = 0 \\ z\Pi_n(z) & if\ n > 0, \end{cases} \tag{2.12}$$

$$\beta_n(z) = \begin{cases} (1-z)\alpha_n(z) = 1 & if\ n = 0 \\ \frac{1-z}{z}(-1)^n\alpha_n(z) & if\ n > 0 \end{cases}. \tag{2.13}$$

Proof. From (2.4) and (2.9) we have

$$\frac{1}{e^t - z} = \sum_{n\geq 0} \frac{\Pi_n(z)}{(1-z)^{n+1}} \frac{(-t)^n}{n!}.$$

Transferring t to $-t$, we change the above equation as

$$\frac{e^t}{1 - ze^t} = \sum_{n\geq 0} \frac{\Pi_n(z)}{(1-z)^{n+1}} \frac{t^n}{n!}.$$

The left-hand side of the above equation can be expanded as

$$\frac{e^t}{1 - ze^t} = \sum_{\ell\geq 0} z^\ell e^{(\ell+1)t} = \sum_{\ell\geq 0}\sum_{n\geq 0} (\ell+1)^n z^\ell \frac{t^n}{n!}.$$

Comparing the right-hand sides of the last two equations yields

$$\frac{\Pi_n(z)}{(1-z)^{n+1}} = \sum_{\ell\geq 0} (\ell+1)^n z^\ell. \tag{2.14}$$

In the introduction, the definitions of Eulerian polynomials and Eulerian fractions are given in (1.1) and (1.4), respectively. Comparing (1.1) and (2.14), we obtain (2.12). Therefore, (2.13) is implied by using (2.12) in the comparison of (1.4) and (2.9) and noting $\beta_n(z) = A_n(0; z)$.

■

From Theorem 2.1, we know that $\Pi_0(z) = 1 = E_0(z)$; $\Pi_1(z) = 1, E_1(z) = z$; $\Pi_2(z) = 1 + z, \quad E_2(z) = z + z^2$, etc.

Remark 2.1 Some immediate results can be found from Theorem 2.1. For examples, (2.14) implies

$$\Pi_{n+1}(z) = (1 + nz)\Pi_n(z) + z(1 - z)\Pi_n'(z). \tag{2.15}$$

Consequently, from (2.12) we have

$$A_{n+1}(z) = (n + 1)zA_n(z) = z(1 - z)A_n'(z),$$

and from (1.4)

$$\alpha_{n+1}(z) = z\alpha_n'(z) - (n + 1)\alpha_n(z).$$

Using Theorem 2.1, we can obtain many (well-known or non-well-known) properties of Eulerian polynomials, Eulerian numbers, and Eulerian fractions from the properties of Euler-Frobenius polynomials, B-splines, and EEPs and vice versa. Here are few examples.

Corollary 2.2 *Eulerian polynomials can be defined by*

$$A_n(z) = n! \sum_{j=1}^{n} B_{n+1}(j) z^j,$$

where $B_n(x)$ denotes the cardinal forward B-spline of order n.

Proof. This is a straight result from Theorem 2.1 and expression (2.11).

■

Remark 2.2 It is well-known that cardinal forward B-spline $B_n(x)$ satisfies

$$B_{n+1}(k) = \frac{n - k + 1}{n} B_n(k - 1) + \frac{k}{n} B_n(k),$$

or equivalently,

$$B_{n+1}(k) = \frac{1}{n!} \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} (k - j)_+^n = \frac{1}{n!} \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k - j)^n \tag{2.16}$$

for $1 \le k \le n$, where $(k - j)_+^0 = 1$ if $k \ge j$ and 0 otherwise, and $(k - j)_+^{m-1} = (k - j)^{m-1}$ if $k \ge j$ and 0 otherwise and the first equation can be seen in P. 11 of [10] or P. 135 of [14]. Formula (2.16) can also be derived from two expressions of

$\Pi_n(z)$ shown as in (2.9) and (2.14). In fact, multiplying $(1-z)^{n+1}$ on both sides of (2.14) yields

$$\begin{aligned}
&\Pi_n(z) = (1-z)^{n+1}\sum_{\ell\geq 0}(\ell+1)^n z^\ell \\
= &\sum_{\ell\geq 0}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}(\ell+1)^n z^{\ell+k} = \sum_{j\geq 0}\sum_{\ell=0}^{j}(-1)^{j-\ell}\binom{n+1}{j-\ell}(\ell+1)^n z^j \\
= &\sum_{j\geq 0}\sum_{\ell=0}^{j+1}(-1)^\ell\binom{n+1}{\ell}(j-\ell+1)^n z^j.
\end{aligned}$$

Comparing the coefficients of z^j on rightmost side of the last equation and the right-hand side of (2.11), we obtain

$$B_{n+1}(j+1) = \frac{1}{n!}\sum_{\ell=0}^{j+1}(-1)^\ell\binom{n+1}{\ell}(j+1-\ell)^n, \quad 0\leq j\leq n,$$

i.e., (2.16) when $k=j+1$, and

$$B_{n+1}(j+1) = \frac{1}{n!}\sum_{\ell=0}^{j+1}(-1)^\ell\binom{n+1}{\ell}(j+1-\ell)^n = \frac{1}{n!}\,\Delta^{j+1}t^n\Big|_{t=0} = 0$$

for all $j\geq n$.

Corollary 2.3 *The generating functions of $\{A_n(z)\}$ and $\{\alpha_n(z)\}$ are respectively*

$$\frac{(1-z)ze^{t(1-z)}}{1-ze^{t(1-z)}} = \sum_{n\geq 0}A_n(z)\frac{t^n}{n!} \tag{2.17}$$

$$\frac{1}{1-ze^t} = \sum_{n\geq 0}\alpha_n(z)\frac{t^n}{n!}. \tag{2.18}$$

Proof. Using transform $t\mapsto t(z-1)$ into (2.4) and, then, substituting relations (2.9) and (2.12) successively in the resulting equation, we have (2.17).

Similarly, substituting (2.13) into (2.4) yields (2.18).

■

Remark 2.3 From (2.17), we can show formula [5i] of [3] readily. By using transform $t\mapsto t(z-1)$ into (2.4) and then substituting relations (2.9) and (2.12) in the resulting equation we obtain

$$1+\frac{1}{z}\sum_{n\geq 1}A_n(z)\frac{t^n}{n!} = \frac{z-1}{z-e^{t(z-1)}}.$$

Hence,

$$\sum_{n\geq 0}A_n(z)\frac{t^n}{n!} = \frac{(z-1)z}{z-e^{t(z-1)}} - z + 1 = \frac{1-z}{1-z^{t(1-z)}},$$

which gives formula [5i] on page 244 of [3].

Corollary 2.4 *Let Eulerian fractions $\alpha_n(z)$ be defined by (1.4). Then $\beta_n(z)$ can be evaluated recursively by using the relation*

$$\alpha_0(z)=\frac{1}{1-z}, \quad (-1)^n z\alpha_n(z)=\frac{z}{1-z}+\sum_{k=1}^{n}(-1)^k\binom{n}{k}\alpha_k(z) \tag{2.19}$$

for $n>0$.

Proof. Obviously, (2.19) is the result of the substitution of (2.13) into (2.7).

∎

Let $z\neq 0,1$. It is well-known that the Eulerian fraction defined by (1.4) can be written as

$$\alpha_n(z)=\sum_{j=0}^{n} j!S(n,j)\frac{z^j}{(1-z)^{j+1}}, \tag{2.20}$$

where $S(n,j)$ are the Stirling numbers of the second kind, i.e., $j!S(n,j)=\left[\Delta^j t^n\right]_{t=0}$, or equivalently, the number of partitions of n distinct elements in j blocks. Hence, from (2.13) and the above formula we have an expression in terms of $\beta_n(z)$ accordingly.

Corollary 2.5 *Let $\beta_n(z)$ be the Euler-Frobenius fractions defined by (2.4). Then*

$$\beta_n(z)=(-1)^n\sum_{j=0}^{n} j!S(n,j)\frac{z^{j-1}}{(1-z)^j}, \tag{2.21}$$

where $S(n,j)$ are the Stirling numbers of the second kind.

Thus, from (2.2), the Euler polynomial of degree n defined by (2.3) for $z=-1$ can be written as

$$A_n(x;-1)=n!2^n\sum_{-\infty}^{\infty}(-1)^j B_{n+1}(x-j).$$

We shall give other expressions of Euler polynomials in terms of Eulerian fractions and Eulerian polynomial values by using Theorem 2.1.

Corollary 2.6 *Let $A_n(x;-1)$ be the Euler polynomial of degree n defined by expansion (2.3). Then it can be expressed as*

$$\begin{aligned} A_n(x;-1) &= \sum_{j=0}^{n}\binom{n}{j}(-1)^{j+1}\alpha_j(-1)x^{n-j}=\sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^{j+1}}{2^j}A_j(-1)x^{n-j} \\ &= \sum_{j=0}^{n}\sum_{k=1}^{j}\binom{n}{j}\frac{(-1)^{j+k+1}j!}{2^j}B_{j+1}(k)x^{n-j}. \end{aligned}$$

Proof. The first expression is from (2.9) for $z=-1$ and (2.21) in Theorem 2.1. By substituting (2.18) for $z=-1$ into the first expression, we obtain the second expression, from which the rightmost equation is derived by using (2.13) for $z=-1$ and (2.20) in Theorem 2.1.

∎

Using (2.12) of Theorem 2.1 in the comparison between (1.2) and (2.11), we immediately have the following corollary (see [20] in a different approach).

Corollary 2.7 *(also see [20]) Let Eulerian numbers $A(n,k)$ be defined by (1.2). Denote the cardinal forward B-spline of order n by $B_n(x)$. Then*

$$A(n,k) = n!B_{n+1}(k) \quad (n > 0). \tag{2.22}$$

Remark 2.4 Since the Eulerian number $A(n,k)$ gives the number of permutations in the symmetric group S_n that have exactly $k-1$ ascents, (2.22) implies that $B_{n+1}(k)$ give the probability of the the event when the permutations in the symmetric group S_n that have exactly $k-1$ ascents. In addition, expression (3) in Chapter 1 in [22] shows that the values of B-spline functions can be defined as the $n!$ multiple of the volumes of slices of a n-dimensional cube. Hence, from the translation formula (2.22), one can immediately give another proof of the result shown in [23] as well as other geometric explanation of Eulerian numbers.

Furthermore, some well-known identities such as [5e], [5e'], and [5h], listed on pages 242-243 in [3] and cited below:

$$\begin{aligned}
&A(n,k) = (n-k+1)A(n-1,k-1) + kA(n-1,k),\\
&A(n,k) = A(n,n-k+1),\\
&x^n = \sum_{k=1}^{n} A(n,k)\binom{x+k-1}{n},
\end{aligned}$$

can be proved trivially using the corresponding identities of B-splines and (2.22). For instance, the second equation of (2.16) implies the first formula. From the symmetry of the B-spline values about the midpoint of the spline's support, one may have the second formula. The third one is also obvious from a well-known property of B-splines.

3 Applications in B-spline image interpolation and computational number theory

The essential property of B-splines $B_n(x)$ of order n is to provide a basis of the subspace of all continuous or differentiable piecewise polynomial functions of degree n. Image interpolation plays a central role in many applications in image processing and engineerings. B-spline interpolation is a powerful tool in the image interpolation. In the case of equally spaced integer knot point, any piecewise polynomial function $a_{n+1}(x)$ of degree n with C^{n-1} continuity can be presented as

$$a_{n+1}(x) = \sum_{j=-\infty}^{\infty} c_n(j)B_{n+1}(x-j), \tag{3.1}$$

where $a_{n+1}(x)$ is uniquely determined by its B-spline coefficients $\{c_n(j)\}$. For a given positive integer m, let us consider a discrete signal $\{f_m(k)\}$ defined on $k = 0, \pm 1, \pm 2, \ldots$, where $f_m(k) = f(\ell)$ if $k = m\ell$ and 0 otherwise. We now seek the interpolatory function $a_{n+1}(x)$ of the form shown in (3.1) such that

$$f_m(k) = a_{n+1}\left(\frac{k}{m}\right) = \sum_{j=-\infty}^{\infty} c_n(j) b_{n+1,m}(k - jm), \tag{3.2}$$

where $b_{n+1,m}(k)$ are discrete splines of order $n+1$ with respect to $m \in \mathbb{N}$ defined by sampling continuous $B_{n+1}(x)$. More precisely, from [14] (P.135), we have

$$B_n(x) = \sum_{j=0}^{n} \frac{(-1)^j}{(n-1)!}\binom{n}{j}(x-j)_+^{n-1},$$

where $(x-j)_+^0 = 1$ if $x \geq j$ and 0 otherwise, and $(x-j)_+^{m-1} = (x-j)^{m-1}$ if $x \geq j$ and 0 otherwise. Thus, we define $b_{n,m}(k)$ by

$$b_{n,m}(k) \equiv B_n\left(\frac{k}{m}\right) = \frac{1}{m^{n-1}} \sum_{j=0}^{n} \frac{(-1)^j}{(n-1)!}\binom{n}{j}(k-jm)_+^{n-1}. \tag{3.3}$$

In [24], Unser, Aldroubi, and Eden presented a method for the image interpolation using fast B-spline transform as follows. The key steps are to take the Z-transform to (3.2) as

$$F(z^m) = B_{n+1,m}(z) C_n(z^m),$$

where $B_{n+1,m}(z)$ is the Z-transform of $\{b_{n+1,m}(k)\}_k$, and use a polynomial sequence $\{A^n(z)\}$ in z^{-1} defined by

$$A^n(z) = (1 - z^{-1})^{n+1} \sum_{j=-\infty}^{\infty} j^n z^{-j} \mu(j),$$

where $\mu(k)$ is the step function taking value 1 when $x \geq 0$ and 0 otherwise. However, the computational manner of $A^n(z)$ is lacked in [24]. Actually, transferring z to z^{-1} in $A^n(z)$ yields

$$A^n(z^{-1}) = (1-z)^{n+1} \sum_{j \geq 0} j^n z^j = A_n(z).$$

Thus, $A^n(z)$ is the nth Eulerian polynomial in terms of z^{-1} from a simple observation of (1.1), namely,

$$A^n(z) = A_n(z^{-1}). \tag{3.4}$$

Thus, following the process of [24] and using Eulerian polynomials, we obtain the Z-transform of $\{b_{n+1,m}(k)\}$ as

$$\begin{aligned} B_{n+1,m}(z) &= \frac{1}{m^n} \sum_{k=-\infty}^{\infty} \left(\sum_{j=0}^{n+1} \frac{(-1)^j}{n!}\binom{n+1}{j}(k-jm)_+^n \right) z^{-k} \\ &= \frac{1}{m^n} \sum_{j=0}^{n+1} \frac{(-1)^j}{n!}\binom{n+1}{j} z^{-jm} \sum_{k=-\infty}^{\infty} k^n z^{-k} \mu(k) \\ &= \frac{A_n(z^{-1})}{(1-z^{-1})^{n+1} m^n} \sum_{j=0}^{n+1} \frac{(-1)^j}{n!}\binom{n+1}{j} z^{-jm} = \frac{1}{m^n} \frac{A_n(z^{-1})}{n!} \left(\frac{1-z^{-m}}{1-z^{-1}} \right)^{n+1}. \end{aligned}$$

From (1.2) and Theorem 2.1, we know that

$$\frac{A_n(z^{-1})}{n!} = \sum_{k=1}^{n} \frac{A(n,k)}{n!} z^{-k} = \sum_{k=1}^{n} B_{n+1}(k) z^{-k}.$$

Therefore,

$$B_{n+1,m}(z) = \frac{1}{m^n} \left(\sum_{j=0}^{m-1} z^{-j} \right)^{n+1} \sum_{k=1}^{n} B_{n+1}(k) z^{-k}.$$

Therefore, noting $A_n(z^{-1})/n! = z\Pi_n(z^{-1})/n! = \sum_{k=1}^{n} B_{n+1}(k)z^{-k}$ and denoting $B_{0,m}(z) = \sum_{j=0}^{m-1} z^{-j}$, we obtain the solution of the B-spline image interpolation as

$$C_n(z^m) = n!m^n F(z^m)/(A_n(z^{-1})B_{0,m}^{n+1}(z)) = n!m^n F(z^m)/(z\Pi_n(z^{-1})B_{0,m}^{n+1}(z)). \tag{3.5}$$

Surveying the above result based on [24], we have

Proposition 3.1 *The B-spline image interpolation problem (3.2) has a unique solution $\{c_n(j)\}$ with its Z-transform $C_n(z^m)$ shown in (3.5). In particular, if $m = 1$, (3.2) is the cardinal B-spline interpolation with the solution*

$$C_n(z) = n!F(z)/A_n(z^{-1}), \tag{3.6}$$

or equivalently,

$$c_n(j) = \begin{cases} n! \sum_{\ell=-j}^{0} f_1(-j-\ell-1)e(n,\ell) & \text{if } j \le 0 \\ n! \sum_{\ell=0}^{j-1} f_1(-j-1)e(n,\ell) & \text{if } j \ge 1 \end{cases}, \tag{3.7}$$

where $\{f_1(j)\}_{j=-\infty}^{\infty}$ is the given data set and

$$e(n,\ell) = -\sum_{k\ge 1} A(n,k+1)e(n,\ell-k). \tag{3.8}$$

Proof. Let $1/A_n(z) = (1/z)\sum_{\ell\ge 0} e(n,\ell)z^\ell$. From

$$1 = \frac{1}{z} A_n(z) \sum_{\ell\ge 0} e(n,\ell) z^\ell = \sum_{k\ge 0} A(n,k+1) z^k \sum_{\ell \ge 0} e(n,\ell) z^\ell,$$

and noting $A(n,1) = 1$, we obtain (3.8) for the expressions of coefficients of reciprocal series of $1/A_n(z^{-1})$. And (3.6) suggests

$$\begin{aligned} & C_n(z) \\ = \ & n!zF(z) \sum_{\ell\ge 0} e(n,\ell) z^{-\ell} = n! \sum_{j=-\infty}^{\infty} f_i(j-1) z^{-j} \sum_{\ell \ge 0} e(n,\ell) z^{-\ell} \\ = \ & n! \left(\sum_{j\ge 0} f_1(j-1) z^{-j} \sum_{\ell\ge 0} e(n,\ell) z^{-\ell} + \sum_{j\ge 1} f_1(-j-1) z^{j} \sum_{\ell\ge 0} e(n,\ell) z^{-\ell} \right), \end{aligned}$$

which implies (3.7) by using Cauchy series multiplication.

■

From Theorem 2.1 and (2.11), we have

$$\begin{aligned} A_n(z^{-1}) &= z^{-1}\Pi_n(z^{-1}) = n!z^{-1}\sum_{j=0}^{n-1} B_{n+1}(j+1)z^{-j} \\ &= n!\sum_{j=1}^{n} B_{n+1}(j)z^{-j} = n!Z(\{B_{n+1}(j)\}_{j\in\mathbb{Z}}), \end{aligned}$$

where $Z(\{B_{n+1}(j)\}_{j\in\mathbb{Z}})$ is the Z-transform of $\{B_{n+1}(j)\}_{j\in\mathbb{Z}}\}$. Hence, besides the multiple of $n!$, nth order Eulerian polynomial in terms of z^{-1} is the Z-transform of the discrete B-spline.

For $m = 1$, with a slight modification in (3.1) one may obtain the fundamental cardinal spline function presented in [10]:

$$\tilde{a}_n(x) = \sum_{j=-\infty}^{\infty} c_n(j)B_n\left(x + \frac{n}{2} - j\right), \tag{3.9}$$

where the cardinal B-spline basis function has support centered at 0. Using $\tilde{a}_n(x)$ as the interpolation basis functions, we need

$$\sum_{j=-\infty}^{\infty} c_n(j)B_n\left(\frac{n}{2} + k - j\right) = \delta_{k,0}, \quad j \in \mathbb{Z},$$

where $\delta_{k,0}$ is the Kronecker symbol. Taking the Z-transformation on the both sides of the last equation yields

$$\tilde{C}_n(z)\tilde{B}_n(z) = 1,$$

and solve $\tilde{C}_n(z) = 1/\tilde{B}_n(z)$, where

$$\tilde{C}_n(z) = \sum_{j\in\mathbb{Z}} c_n(j)z^{-j}$$

and

$$\tilde{B}_n(z) = \sum_{j\in\mathbb{Z}} B_n\left(\frac{n}{2} + j\right)z^{-j} = \sum_{j\in\mathbb{Z}} B_n\left(\frac{n}{2} + j\right)z^{j}.$$

The last equation comes from the symmetry of $B_n(j + n/2) = B_n(n - j - n/2)$ and the compactness of the support of B_n. We call both expressions of $(n-1)!\tilde{B}_n(z)$ the *fundamental Eulerian polynomial or fundamental Euler-Frobenius polynomial* of order n.

By surveying the above results, we have

Proposition 3.2 *For any $n \in \mathbb{N}$, the Eulerian polynomial $A_n(z^{-1}) = n!\sum_{j\in\mathbb{Z}} B_{n+1}(j)z^{-j}$ is a constant multiple of the Z-transform of $\{B_{n+1}(j)\ \}_{0\le j\le n+1}$, or equivalently, $\hat{A}_n(t) = n!\sum_{j\in\mathbb{Z}} B_{n+1}(j)e^{ijt}$ is the Fourier transform of $\{B_{n+1}(j)\}_{0\le j\le n+1}$ up to a constant multiple. And the fundamental Eulerian polynomials $(n-1)!\sum_{j\in\mathbb{Z}} B_n(j+n/2)z^j$ is a constant multiple of the Z-transform of $\{B_n(j+n/a)\}_{j\in\mathbb{Z}}$.*

From [25], one may find the fundamental Eulerian polynomials are strictly positive. [26] presents an extension of the fundamental Eulerian polynomials to the higher dimensional setting:

$$P_\Xi(x) := \sum_{j\in\mathbb{Z}^d} M_\Xi(j)e^{ijx}, \quad x\in\mathbb{R}^d,$$

where M_Ξ is the box-spline generated by vector set $\Xi=\{\xi_1,\ldots,\xi_n\}$, $\xi\in\mathbb{Z}^d$. In the bivariate case ($d=2$), [26] proves the following conjecture: The polynomials P_Ξ are strictly positive iff the box-splines $M_\Xi(\cdot-j)$, $j\in\mathbb{Z}^d$, are linearly independent. If the conjecture is valid in general ($d>2$), then it would imply that fundamental cardinal interpolation is well posed when the obvious necessary condition of linear independence is satisfied. For case $d=2$, [27] shows that the box-splines are linearly independent only on the regular three-direction mesh, i.e., $\Xi=\{(1,0),(0,1),(1,1)\}$. Thus, the fundamental cardinal bivariate box-spline interpolation over the regular three-direction mesh is well posed.

We now consider the application of Euler-Frobenius fractions, $\beta_n(z)$ ($|z|\le 1$), defined by (2.9), in computational number theory. Denote

$$\gamma_n(z)=\frac{2(-1)^n}{1+z}\beta_n\left(-\frac{1}{z}\right), \tag{3.10}$$

where $z\in[1,1+c]$, $c>0$. Thus, from (2.14), we have

$$\begin{aligned}\gamma_n(z)&=\frac{2(-1)^n}{1+z}\beta_n\left(-\frac{1}{z}\right)=\frac{2\Pi_n\left(-\frac{1}{z}\right)}{z\left(1-\left(-\frac{1}{z}\right)\right)^{n+1}}\\ &=\frac{2}{z}\sum_{\ell\ge 0}(\ell+1)^n(-z)^{-\ell}=-2\sum_{\ell\ge 1}\ell^n(-z)^{-\ell}.\end{aligned} \tag{3.11}$$

We can also obtain the generating function of $\{\gamma_n(z)\}$ from (2.4):

$$\sum_{n\ge 0}\gamma_n(z)\frac{t^n}{n!}=\frac{2}{1+z}\sum_{n\ge 0}\beta_n\left(-\frac{1}{z}\right)\frac{(-t)^n}{n!}=\frac{2}{1+z}\frac{-\frac{1}{z}-1}{-\frac{1}{z}-e^{-t}}=\frac{2e^t}{z+e^t}. \tag{3.12}$$

Recall the Euler polynomials $A_n(x;-1)$ defined by (2.3) with $z=-1$, we know that $\gamma_n(1)=A_n(1;-1)$. In addition, it is easy to see $\gamma_{-1}(1)=2\ln 2$ and

$$\gamma_{-n}(1)=-2(2^{1-n}-1)\zeta(n), \tag{3.13}$$

where $\zeta(z)$ is the Riemann zeta function, and $n>1$ is an integer. Finally, for $z\ge 1$,

$$\overline{\lim_{n\to\infty}}\left(\frac{|\phi_n(z)|}{n!}\right)^{1/n}\le\frac{1}{\pi}, \tag{3.14}$$

and thus the series in Eq. (3.12) converges absolutely for $|t|<\pi$.

For any positive integer k, we have

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} \frac{(-u)^{-n}\sin(n\pi)}{n^{2k}} = \sum_{n=1}^{\infty} \frac{(-u)^{-n}}{n^{2k}} \sum_{j=0}^{\infty} (-1)^j \frac{(n\pi)^{2j+1}}{(2j+1)!} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^{j+1} u\pi^{2j+1}}{2(2j+1)!} \gamma_{2j+1-2k}(u) \\ &= \sum_{j=0}^{k-1} \frac{(-1)^{j+1} u\pi^{2j+1}}{2(2j+1)!} \gamma_{2j+1-2k}(u) + \sum_{m=0}^{\infty} \frac{(-1)^{m+k+1} u\pi^{2m+2k+1}}{2(2m+2k+1)!} \gamma_{2m+1}(u). \end{aligned}$$

In light of (3.14), we can now let $u \to 1^+$, obtaining

$$0 = \sum_{j=0}^{k-1} \frac{(-1)^{j+1}\pi^{2j+1}}{2(2j+1)!} \gamma_{2j+1-2k}(1) + \sum_{m=0}^{\infty} \frac{(-1)^{k+1}\pi^{2k+1} f_m}{2(2m+2k+1)!}, \tag{3.15}$$

where $f_m = (-1)^m \pi^{2m} A_{2m+1}(1;-1)$.

Setting $k=1$ in (3.15) and recalling that $\gamma_{-1}(1) = 2\ln 2$, we have the curious formula

$$\ln 2 = \frac{\pi^2}{2} \sum_{m=0}^{\infty} \frac{(-1)^m \pi^{2m}}{(2m+3)!} A_{2m+1}(1;-1). \tag{3.16}$$

We can use (3.13) and (3.15) to deduce the following theorem, which gives $\zeta(2k+1)$ recursively in terms of $\ln 2$, $\zeta(3), \ldots, \zeta(2k-1)$:

Theorem 3.3 *For any positive integer k,*

$$\begin{aligned} (1-2^{-2k})\zeta(2k+1) &= \sum_{j=1}^{k-1} \frac{(-1)^j \pi^{2j}}{(2j+1)!} \left(2^{2j-2k}-1\right) \zeta(2k-2j+1) \\ &- \frac{(-1)^k \pi^{2k} \ln 2}{(2k+1)!} + \frac{(-1)^k \pi^{2k+2}}{2} \sum_{m=0}^{\infty} \frac{(-1)^m \pi^{2m} A_{2m+1}(1;-1)}{(2m+2k+3)!}. \end{aligned} \tag{3.17}$$

From (3.17), an Euler-type formula for $\zeta(2k+1)$ can be constructed (see [28] for more details). Finally, we should mention that there are many interesting applications of Eulerian (Euler-Frobinous) polynomials to spline wavelet analysis, for instance, in [29] and [30].

Acknowledgments

The author wish to thank the editor and referees for their helpful comments and suggestions.

References

[1] L. Carlitz, Eulerian numbers and polynomials of higher order. Duke Math. J. 27 1960 401–423.

[2] L. Carlitz, D. P. Roselle, and R. Scoville, Permutations and sequences with repetitions by number of increase. J. Combin. Th. 1 (1966), 350-374.

[3] L. Comtet, Advanced Combinatorics-The Art of Finite and Infinite expansions, Dordrecht: Reidel, 1974.

[4] J. Riordan, An Introduction to Combinatorial Analysis, John Wiley & Sons, New York, 1958.

[5] M. Aigner, Kombinatorik I, Springer-Verlag, Berlin, 1975.

[6] R. L. Graham, D. E. Knuth, and O. Patashnik, Concrete Mathematics, Addison-Wesley Publishing Company, Reading, Mass., 1989.

[7] M. de Laplace, Oeuvres Complétes, Vol. 7, rèèditè par Gauthier-Villars, Paris, 1886.

[8] R. Ehrenborg and M. Readdy, Mixed volumes and slices of the cube, J. Comb. Theory, Series A, 81 (1998), 121-126.

[9] G. Birkhoff and C. de Boor, Piecewise polynomial interpolation and approximation. 1965 Approximation of Functions (Proc. Sympos. General Motors Res. Lab., 1964) pp. 164–190 Elsevier Publ. Co., Amsterdam.

[10] I. J. Schoenberg, Cardinal Spline Interpolation, CBMS-NSF Regional Conference Series in Applied Mathematics, 1973.

[11] C. de Boor, On the cardinal spline interpolant to e^{iut}. SIAM J. Math. Anal. 7 (1976), no. 6, 930–941.

[12] C. deBoor, Chapter II. Splines with uniform knots, draft jul74, TeXed 1997.

[13] C. de Boor, A practical guide to splines. Revised edition. Applied Mathematical Sciences, 27. Springer-Verlag, New York, 2001.

[14] L. Schumaker, Spline Functions: Basic Theory, John Wiley & Sons, New York, 1981.

[15] O. A. Vasicek and H. G. Fong, Term Structure Modeling Using Exponential Splines, The Journal of Finance, Vol. 37, No. 2, Papers and Proceedings of the Fortieth Annual Meeting of the American Finance Association, Washington, D.C., December 28-30, 1981. (May, 1982), 339-348.

[16] C. Zoppou, S. Roberts and R. J. Renka, Exponential spline interpolation in characteristic based scheme for solving the advective-diffusion equation, International Journal of Numerical Methods in Fluids, 33(3) (2000), 429-452.

[17] N. Dyn and A. Ron, Recurrence relations for Tchebycheffian B-splines. J. Analyse Math. 51 (1988), 118–138.

[18] A. Ron, Exponential box splines. Constr. Approx. 4 (1988), no. 4, 357–378.

[19] T. X. He, Generalized Eulerian fractions, exponential splines, and related topics (Conference talk Abstract), International Conference on Interactions between Wavelets and Splines, Athens, Georgia, May 16, 2005.

[20] R.-H. Wang, Y. Xu, and Z.-Q. Xu, Eulerian numbers: A spline perspective, J. Math. Anal. Appl. 370 (2010), 486-490.

[21] C. de Boor, Personal communication, 2005.

[22] C. de Boor, K. Hllig, and S. Riemenschneider, Box splines. Applied Mathematical Sciences, 98. Springer-Verlag, New York, 1993.

[23] R. P. Stanley, Eulerian partitions of a unit hypercube, in "Higher Combinatorics (Proc. NATO Advanced Study Inst., Berlin, Sept. 1-10, 1976" (Aigner, M., Ed.), p. 49. NATO Adv. Study Inst. Ser., Ser. C: Math. and Phys. Sci., 31. Reidel, Dordrecht, 1977.

[24] M. Unser, A. Aldroubi, and M. Eden, Fast B-spline transforms for continuous image representation and interpolation, IEEE Trans. Pattern Analysis and Machine Intellegence, 13 (1991), No. 3, 277-285.

[25] I. J. Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions. Part A. On the problem of smoothing or graduation. A first class of analytic approximation formulae. Quart. Appl. Math. 4, (1946). 45–99.

[26] C. de Boor, K. Hllig, and S. Riemenschneider, Bivariate cardinal interpolation by splines on a three-direction mesh. Illinois J. Math. 29 (1985), no. 4, 533–566.

[27] C. de Boor and K. Hllig, Bivariate box splines and smooth pp functions on a three direction mesh. J. Comput. Appl. Math. 9 (1983), no. 1, 13–28.

[28] M. Dancs and T. X. He, An Euler-type formula for $\zeta(2k+1)$, J. Number Theory, 118 (2006), 192-199.

[29] C. K. Chui, An Introduction to Wavelets, Academic Press, Inc., New York, 1992.

[30] S. L. Lee, A. Sharma, and H. H. Tan, Spline interpolation and wavelet construction, Appl. Comp. Harmonic Anal. 5 (1998), 249-276.